SELF-DUAL REPRESENTATIONS WITH VECTORS FIXED UNDER AN IWAHORI SUBGROUP

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ABSTRACT

Let G be the group of F-points of a split connected reductive F-group over a non-Archimedean local field F of characteristic 0. Let π be an irreducible smooth self-dual representation of G. The space W of π carries a non-degenerate G-invariant bilinear form $(\,,\,)$ which is unique up to scaling. The form is easily seen to be symmetric or skew-symmetric and we set $\varepsilon(\pi) = \pm 1$ accordingly. In this article, we show that $\varepsilon(\pi) = 1$ when π is a generic representation of G with non-zero vectors fixed under an Iwahori subgroup I.

INTRODUCTION

Let G be a group and (π, V) be an irreducible representation of G such that $\pi \simeq \pi^\vee$ ($\pi^\vee$ denotes the dual or contragredient of π). This isomorphism gives rise to a non-degenerate G-invariant bilinear form on V which is unique up to scalars, and consequently is either symmetric or skew-symmetric. Accordingly, we set

$$\varepsilon(\pi) = \begin{cases} 1 & \text{if the form is symmetric} \\ -1 & \text{if the form is skew-symmetric} \end{cases}$$

and call it the sign of π. In this article, we study this sign for a certain class of representations of a reductive p-adic group G. To be more precise, we study the sign for representations with non-zero vectors fixed under an Iwahori subgroup in G. The structure of representations with Iwahori fixed vectors is well understood and this is one of the principal reasons we restrict our analysis to this particular class of representations.

OVERVIEW OF THE PROBLEM

Let G be a group and (π, V) be an irreducible self-dual complex representation of G. We say that the representation π is realizable over the real numbers if there exists a G-invariant real subspace W of V such that

$V \simeq \mathbb{C} \otimes_{\mathbb{R}} W$ as representations of G. A classical problem in representation theory was to determine when such a W exists. For a finite group G, this problem was settled by Frobenius and Schur more than a century ago. They showed that π is realizable over the real numbers precisely when $\varepsilon(\pi) = 1$. They also gave a formula to compute $\varepsilon(\pi)$ in terms of the character χ_π of the representation π. They showed that

$$\varepsilon(\pi) = \frac{1}{|G|} \sum_{g \in G} \chi_\pi(g^2).$$

The sign $\varepsilon(\pi)$ has been fairly extensively studied for connected compact Lie groups and finite groups of Lie type. In this setting, the sign is sometimes referred to as the Frobenius-Schur indicator of π. There is a lot of literature available on computing these signs for such groups. For connected compact Lie groups the sign can be computed using the dominant weight attached to the representation π (see [2] pg. 261-264). In [6], Gow showed that for q a power of an odd prime and F_q the finite field with q elements, irreducible self-dual complex representations of $\mathrm{SO}(n, F_q)$ are always realizable over $\mathbb{R}$. He also showed that the same is true for any non-faithful representation of $\mathrm{Sp}(n, F_q)$. The proofs involve a detailed analysis of the conjugacy classes of these groups and are computationally quite complicated. In [8], Prasad introduced an elegant idea to compute the sign for a certain class of representations of finite groups of Lie type. These representations are called generic. He used this idea to determine the sign for many finite groups of Lie type, avoiding tedious conjugacy class computations. In a subsequent paper [9], he extended this idea to representations of a reductive p-adic group G and computed the sign for generic representations of certain classical groups in some cases. In [12], Vinroot used Prasad's idea along with other techniques to compute the sign for an irreducible self-dual representation of $\mathrm{GL}(n, F)$ where F is a p-adic field.

In this article, we determine the sign $\varepsilon(\pi)$ when π is a certain type of representation of an arbitrary connected reductive p-adic group G which is split over the underlying p-adic field F. Suppose K is the maximal compact open subgroup of G and π has non-zero vectors fixed under K. In this situation, it is easy to see that $\varepsilon(\pi)$ is always 1. A natural question is what is $\varepsilon(\pi)$ if π has non-zero Iwahori fixed vectors. There is enough evidence that the sign is one in the Iwahori fixed case. We have not been able to prove the result in complete generality. However, we do address a particular case of this problem. To be more precise, we prove the following theorem.

Theorem 0.1 (Main Theorem)**.** *Let (π, W) be an irreducible smooth self-dual representation of G with non-zero vectors fixed under an Iwahori subgroup in G. Suppose that π is also generic. Then $\varepsilon(\pi) = 1$.*

1. Self-dual representations and signs

In this section, we introduce and briefly discuss the notion of signs associated to self-dual representations.

1.1. Sign of π.

Let F be a non-Archimedean local field and G be the group of F-points of a connected reductive algebraic group. Let (π, W) be a smooth irreducible representation of G. We write $(\pi^\vee, W^\vee)$ for the smooth dual or contragredient of (π, W) and $\langle \, , \, \rangle$ for the canonical non-degenerate G-invariant pairing on $W \times W^\vee$ (given by evaluation). Let $s : (\pi, W) \to (\pi^\vee, W^\vee)$ be an isomorphism. The map s can be used to define a bilinear form on W as follows

$$(w_1, w_2) = \langle w_1, s(w_2) \rangle, \quad \forall w_1, w_2 \in W.$$

It is easy to see that $(\, , \,)$ is a non-degenerate G-invariant form on W, i.e., it satisfies,

$$(\pi(g)w_1, \pi(g)w_2) = (w_1, w_2), \quad \forall w_1, w_2 \in W.$$

Let $(\, , \,)_*$ be a new bilinear form on W defined by

$$(w_1, w_2)_* = (w_2, w_1)$$

Clearly, this form is again non-degenerate and G-invariant. It follows from Schur's Lemma that

$$(w_1, w_2)_* = c(w_1, w_2)$$

for some non-zero scalar c. A simple computation shows that $c \in \{\pm 1\}$. Indeed,

$$(w_1, w_2) = (w_2, w_1)_* = c(w_2, w_1) = c(w_1, w_2)_* = c^2(w_1, w_2).$$

We set $c = \varepsilon(\pi)$. It clearly depends only on the equivalence class of π. In sum, the form $(\, , \,)$ is symmetric or skew-symmetric and the sign $\varepsilon(\pi)$ records its type.

2. Representations of some classical groups

We use a theorem of Waldspurger to show that many representations of classical groups are self-dual. Throughout this section, we let F be a non-Archimedean local field of characteristic $\neq 2$ and W be a finite dimensional vector space over F. We write $\mathfrak{O}$ for the ring of integers in F, $\mathfrak{p}$ for the unique maximal ideal of $\mathfrak{O}$ and k for the residue field. We let $\langle \, , \, \rangle$ to be a non-degenerate symmetric or skew-symmetric form on W. We take

$$G = \{ g \in \mathrm{GL}(W) \mid \langle gw, gw' \rangle = \langle w, w' \rangle \}$$

For $x \in \mathrm{GL}(W)$ such that $xGx^{-1} = G$ and (π, V) a representation of G, we let π^x denote the representation of G defined by conjugation (i.e., $\pi^x(g) = \pi(xgx^{-1})$).

We recall the statement of Waldspurger's theorem and refer the reader to Chapter 4.II.1 in [7] for a proof.

Theorem 2.1 (Waldspurger). *Let π be an irreducible admissible representation of G and $\pi^\vee$ be the smooth-dual or contragredient of π. Let $x \in \mathrm{GL}(W)$ be such that $\langle xw, xw' \rangle = \langle w', w \rangle,\, \forall\, w, w' \in W$. Then $\pi^x \simeq \pi^\vee$.*

Proof. See chapter 4.II.1 in [7]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.1. **Orthogonal groups.** Suppose the form $\langle\,,\,\rangle$ is symmetric so that G is the orthgonal group $\mathrm{O}(W)$. Let π be any irreducible admissible representation of G. Then $x = 1 \in G$ satisfies $\langle xw, xw' \rangle = \langle w', w \rangle, \forall\, w, w' \in W$. Now using Waldspurger's Theorem, it follows that $\pi \simeq \pi^\vee$. So in the case of orthogonal groups every irreducible representation π is self-dual.

2.2. **Special orthogonal groups.** Suppose the dimension of W is odd. Take $G = \mathrm{SO}(W) = \mathrm{O}(W) \cap \mathrm{SL}(W)$ and π to be an irreducible admissible representation of G. Since $\mathrm{O}(W) \simeq \mathrm{SO}(W) \times \{\pm 1_W\}$, it follows that there exists an irreducible representation $\tilde{\pi}$ of $\mathrm{O}(W)$ such that $\tilde{\pi} \simeq \pi \otimes \chi$ (where χ is a character of $\{\pm 1_W\}$). Since $\chi = \chi^{-1}$ and $\tilde{\pi} \simeq \tilde{\pi}^\vee$ it follows that $\pi \simeq \pi^\vee$.

2.3. **Symplectic groups.** Suppose that G is the symplectic group $Sp(W)$ or $Sp(n, F)$. We show that a certain class of representations of G is always self-dual. To be more precise, we prove,

Theorem 2.2. *Let (π, V) be an irreducible admissible representation of G with non-zero vectors fixed under an Iwahori subgroup I in G. Then $\pi \simeq \pi^\vee$.*

Consider $x = \begin{bmatrix} -\mathrm{I} & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}(W)$ (where I is the $n \times n$ identity matrix). It is easy to see that $\langle xw, xw' \rangle = \langle w', w \rangle$. By Theorem 2.1, $\pi^x \simeq \pi^\vee$. To prove $\pi \simeq \pi^\vee$, it suffices to show that $\pi \simeq \pi^x$. Observe that $xIx^{-1} = I$ and $\pi^I = (\pi^x)^I$. Since $\pi^I = (\pi^x)^I \neq 0$, they can be realized as simple modules over $\mathcal{H}(G, I)$ (where $\mathcal{H}(G, I)$ is the Iwahori-Hecke algebra). Let $f \bullet v$ and $f \star v$ denote the action of $\mathcal{H}(G, I)$ on π^I and $(\pi^x)^I$ respectively. It will follow that $\pi \simeq \pi^x$ if π^I and $(\pi^x)^I$ are equivalent as $\mathcal{H}(G, I)$-modules. We establish this equivalence, by showing that the map $\phi = 1_V$ (identity map on V) defines an intertwining map between π^I and $(\pi^x)^I$.

Before we continue, we fix a collection of coset representatives for the affine Weyl group $\widetilde{\mathcal{W}}$ and record two lemmas we need.

Let $B_{i,i+1}, i = 1, 2, \ldots, n-1$, be the $n \times n$ matrix

$$
B_{i,i+1} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \left.\begin{matrix}\\ \\ \end{matrix}\right\} \begin{matrix} i \\ i+1 \end{matrix} \\ \\ \\ \end{matrix}
$$

and w_n and w_ℓ, $\ell \in \mathbb{Z}^n$, be the following $2n \times 2n$ matrices

$$
w_n = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 1 \end{bmatrix} \left. \begin{array}{c} \\ \\ \\ \\ \end{array} \right\} \begin{array}{c} n \\ n+1 \end{array} \;,
$$

$$
w_\ell = \begin{bmatrix} \varpi^{l_1} & & & & & \\ & \ddots & & & & \\ & & \varpi^{l_n} & & & \\ & & & \varpi^{-l_n} & & \\ & & & & \ddots & \\ & & & & & \varpi^{-l_1} \end{bmatrix} , l_i \in \mathbb{Z}.
$$

The group $\left\langle w_\ell, w_n, w_i = \begin{bmatrix} B_{i,i+1} & 0 \\ 0 & B_{i,i+1} \end{bmatrix} \middle| \ell \in \mathbb{Z}^n, i = 1, 2, \dots, n-1 \right\rangle$ contains a collection of coset representatives for the affine Weyl group $\widetilde{\mathcal{W}}$.

Lemma 2.3. *For $f \in \mathcal{H}(G, I)$, let $f^x \in \mathcal{H}(G, I)$ be the function $f^x(g) = f(x^{-1}gx)$. The following statements are true.*

    (i) $f \star v = f^x \bullet v$.
    (ii) *For $g = i_1 w i_2 \in G$ (Bruhat Decomposition), $f^x(g) = f(xwx^{-1})$ and $f(g) = f(w)$.*

Proof. Clearly $f \star v = f^x \bullet v$. Indeed,

$$
\begin{aligned}
f \star v &= \int_G f(g) \pi^x(g) v \, dg \\
&= \int_G f(g) \pi(xgx^{-1}) v \, dg \\
&= \int_G f(x^{-1}gx) \pi(g) v \, dg \\
&= f^x \bullet v.
\end{aligned}
$$

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For (ii),

$$
\begin{aligned}
f^x(g) &= f(xi_1 w i_2 x^{-1}) \\
&= f(xi_1 x^{-1} x w x^{-1} x i_2 x^{-1}) \\
&= f(i x w x^{-1} i') \\
&= f(x w x^{-1}),
\end{aligned}
$$

and

$$
\begin{aligned}
f(g) &= f(i_1 w i_2) \\
&= f(w).
\end{aligned}
$$

$\square$

Lemma 2.4. *For $w \in \widetilde{W}$, $x w x^{-1} \in IwI$.*

Proof. If $w \in \widetilde{W}$, we can write $w = u_1 u_2 \ldots u_n$ for $u_k \in \{w_\circ, w_i, w_\ell\}$. In this case we say w has length n and denote it as $l(w) = n$. We will use induction on the length $l(w)$ to show that $x w x^{-1} \in IwI$. Suppose that $l(w) = 1$. A simple computation shows that conjugation by the element x fixes the elements w_i, w_ℓ and fixes w_n up to multiplication by elements in $T_\circ \subset I$, i.e., $x w_i x^{-1} = w_i$ for $i = 1, \ldots, n-1$, $x w_\ell x^{-1} = w_\ell$ for $\ell \in \mathbb{Z}^n$ and $x w_n x^{-1} = t w_n t^{-1}$ for some $t \in T_\circ \subset I$. Suppose $l(w) = 2$, i.e., $w = u_1 u_2$ for $u_1, u_2 \in \{w_0, w_i, w_\ell\}$ such that $x u_1 x^{-1} = t_1 u_1 t_1^{-1}$ and $x u_2 x^{-1} = t_2 u_2 t_2^{-1}$, $t_1, t_2 \in T_\circ$. In this case, we have

$$
\begin{aligned}
x w x^{-1} &= x u_1 x^{-1} x u_2 x^{-1} \\
&= t_1 u_1 t_1^{-1} t_2 u_2 t_2^{-1} \\
&= t_1 u_1 u_2 \underbrace{u_2^{-1} t_1^{-1} u_2}_{\in T_\circ} \underbrace{u_2^{-1} t_2 u_2}_{\in T_\circ} t_2^{-1} \\
&= t_1 u_1 u_2 t_1'
\end{aligned}
$$

where $t, t' \in T_\circ$.

Assume that the result is true for all words w such that $l(w) \leq n - 1$. Suppose $w = u_1 u_2 \ldots u_n$. Now

$$
\begin{aligned}
x w x^{-1} &= x u_1 u_2 \ldots u_{n-1} x^{-1} x u_n x^{-1} \\
&= t u_1 u_2 \ldots u_{n-1} t^{-1} x u_n x^{-1} \\
&= t w t' \quad \text{(by previous case)}
\end{aligned}
$$

where $t, t' \in T_\circ$. $\square$

We now prove Theorem 2.2. To prove $\phi = 1_V$ is an intertwining map, we need to show that $f \bullet v = f \star v = f^x \bullet v$. By Lemma 2.3, it suffices to show that $f(x w x^{-1}) = f(w)$, for all $w \in \widetilde{\mathcal{W}}$. By Lemma 2.4, it follows that conjugation by the element x fixes every element in $\widetilde{W}$ (up to multiplication

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by elements in I). The result now follows.

3. Results used in proof of main theorem

In this section, we recall the important results used in the proof of the main theorem.

3.1. **Restriction of representations to subgroups.** We recall some results about restricting an irreducible representation to a subgroup. These results hold when G is a locally compact totally disconnected group and H is an open normal subgroup of G such that G/H is finite abelian. For a more detailed account, we refer the reader to [5] (Lemma 2.1, 2.3).

Theorem 3.1 (Gelbart-Knapp)**.** *Let π be an irreducible admissible representation of G. Suppose that G/H is finite abelian. Then*

(i) *$\pi|_H$ is a finite direct sum of irreducible admissible representations of H.*

(ii) *When the irreducible constituents of $\pi|_H$ are grouped according to their equivalence classes as*

$$\pi|_H = \bigoplus_{i=1}^{M} m_i \pi_i$$

with the π_i irreducible and inequivalent, the integers m_i are equal.

Theorem 3.2 (Gelbart-Knapp)**.** *Let G be a locally compact totally disconnected group and H be an open normal subgroup of G such that G/H is finite abelian, and let π be an irreducible admissible representation of H. Then*

(i) *There exists an irreducible admissible representation $\tilde{\pi}$ of G such that $\tilde{\pi}|_H$ contains π as a constituent.*

(ii) *Suppose $\tilde{\pi}$ and $\tilde{\pi}'$ are irreducible admissible representations of G whose restrictions to H are multiplicity free and contain π. Then $\tilde{\pi}|_H$ and $\tilde{\pi}'|_H$ are equivalent and $\tilde{\pi}$ is equivalent with $\tilde{\pi}' \otimes \chi$ for some character χ of G that is trivial on H.*

3.2. **Unramified principal series and representations with Iwahori fixed vectors.** We define the notion of an unramified principal series representation and state an important characterization of representations with non-zero vectors fixed under an Iwahori subgroup due to Borel and Casselman. We refer to ([1], [4]) for a proof.

Throughout this section, we let G be the group of F-points of a connected reductive algebraic group defined and split over F. We write T for a maximal F-split torus in G. We also fix a Borel subgroup B defined over F such that $B \supset T$ and write U for the unipotent radical of B.

Let (ρ, W) be a smooth representation of T. We can view ρ as a smooth representation of B which is trivial on U, and form the smooth induced representation $\mathrm{Ind}_B^G \rho$. The representations $\mathrm{Ind}_B^G(\mu)$, where μ is an unramified character of T (i.e., $\mu|_{T_\circ} = 1$ where $T_\circ$ is the $\mathfrak{O}$-points of T) are called the unramified principal series representations.

Theorem 3.3 (Borel-Casselman). *Let (π, W) be any irreducible admissible representation of G. Then the following assertions are equivalent.*

   (i) *There are non-zero vectors in W invariant under I.*
   (ii) *There exists some unramified character μ of T such that π imbeds as a sub-representation of $\mathrm{Ind}_B^G \mu$.*

3.3. **Prasad's idea for computing the sign.** In [9], Prasad gives a criterion to compute the sign for an irreducible self-dual generic representation of a p-adic group G. He shows that for generic representations the sign is determined by the value of the central character ω_π at a special central element. We recall his result below.

Theorem 3.4 (Prasad). *Let K be a compact open subgroup of G. Let s be an element of G which normalizes K and whose square belongs to the center of G. Let $\psi_K : K \to \mathbb{C}^\times$ be a one dimensional representation of K which is taken to its inverse by inner conjugation action of s on K. Let π be an irreducible representation of G in which the character ψ_K of K appears with multiplicity 1. Then if π is self-dual, $\varepsilon(\pi)$ is 1 if and only if the element s^2 belonging to the center of G operates by 1 on π.*

3.4. **Compact approximation of Whittaker models.** In this section, we recall an important result of Rodier which we need in the proof of the main theorem. We refer the reader to [10] for a more detailed account of Rodier's results.

Theorem 3.5 (Rodier). *Let π be an irreducible admissible representation of G and ψ be a non-degenerate character of U. There then exists a compact open subgroup K and a character ψ_K of K such that*

$$dim_\mathbb{C} \mathrm{Hom}_K(\pi, \psi_K) = dim_\mathbb{C} \mathrm{Hom}_U(\pi, \psi).$$

Therefore, if π is generic, $dim_\mathbb{C} \mathrm{Hom}_K(\pi, \psi_K) = 1$.

4. Main Theorem

In this section, we prove the main theorem. We recall the statement below.

Theorem 4.1 (Main Theorem). *Let (π, W) be an irreducible smooth self-dual representation of G with non-zero vectors fixed under an Iwahori subgroup I in G. Suppose that π is also generic. Then $\varepsilon(\pi) = 1$.*

We first prove the result when G has connected center. In this case, we use a result of Rodier (Theorem 3.5) to get a compact open subgroup K and

a character ψ_K of K which appears with multiplicity one in $\pi|_K$. We show that there exists an element $s \in T$ satisfying the hypotheses of Prasad's Theorem (Theorem 3.4). Finally we use the fact that π has non-zero Iwahori fixed vectors to show that $\varepsilon(\pi) = 1$.

When the center of G is not connected, we construct a split connected reductive F-group $\tilde{G}$ with a maximal F-split torus $\tilde{T}$. The group $\tilde{G}$ has a connected center $\tilde{Z}$ and contains G as a subgroup. We show that there exists an irreducible representation $\tilde{\pi}$ of $\tilde{G}$ that contains the representation π with multiplicity one on restriction to G and has non-zero vectors fixed under an Iwahori subgroup in $\tilde{G}$. The representation $\tilde{\pi}$ is not necessarily self-dual but is self-dual up to a twist by a character χ of $\tilde{G}$ which is trivial on G. We can still attach a sign $\varepsilon(\tilde{\pi})$ to $\tilde{\pi}$. Finally, we show that $\varepsilon(\tilde{\pi}) = \varepsilon(\pi)$ and $\varepsilon(\tilde{\pi}) = 1$.

Throughout this section, we let G be the group of F-points of a connected reductive algebraic group defined and split over F. We write T for a maximal F-split torus in G. We also fix a Borel subgroup B defined over F such that $B \supset T$. We write U for the unipotent radical of B (respectively $\bar{U}$ for the T-opposite of U) and fix a non-degenerate character ψ of U such that $\mathrm{Hom}_U(\pi, \psi) \neq 0$ (ψ exists since π is generic). We let X and $X^\vee$ be the character and cocharacter groups of T. We write Φ and $\Phi^\vee$ for the set of roots and coroots and Δ for the set of simple roots of T. Since T is F-split, we have unique subgroups $T_\circ$ and T_1 of T such that $T = T_\circ \times T_1$. To be more precise, the isomorphism $F^\times \simeq \mathfrak{O}^\times \times \mathbb{Z}$ given by $x\varpi^n \mapsto (x, n)$ induces the following isomorphism

$$T \simeq X^\vee \otimes F^\times \simeq X^\vee \otimes \mathfrak{O}^\times \oplus X^\vee \otimes \mathbb{Z}$$

and we take $T_\circ$ and T_1 to be the subgroups of T such that $T_\circ \simeq X^\vee \otimes \mathfrak{O}^\times$ ($\alpha^\vee \otimes y \to \alpha^\vee(y)$) and $T_1 \simeq X^\vee \otimes \mathbb{Z}$ ($\alpha^\vee \otimes n \to \alpha^\vee(\varpi^n)$). We have a similar decomposition for $\tilde{T}$ (i.e., $\tilde{T} = \tilde{T}_\circ \times \tilde{T}_1$). In what follows, we let $Z_\circ = Z \cap T_\circ$ and $Z_1 = Z \cap T_1$ (respectively $\tilde{Z}_\circ = \tilde{Z} \cap \tilde{T}_\circ$ and $\tilde{Z}_1 = \tilde{Z} \cap \tilde{T}_1$). We let $\omega_\circ = \omega_\pi|_{Z_\circ}$ and $\omega_1 = \omega_\pi|_{Z_1}$.

4.1. **Center of G is connected.**

. In this section, we show the existence of an element $s \in T$ satisfying the conditions of Prasad's theorem (Theorem 3.4) and use it to compute the sign $\varepsilon(\pi)$ when the center of G is connected.

Lemma 4.2. *Let $s \in T$ be such that $\alpha(s) = -1$ for all simple roots $\alpha \in \Delta$. The following are true.*

   (i) *For $u \in U$, we have $\psi(sus^{-1}) = \psi^{-1}(u)$.*
   (ii) *The element s^2 belongs to the center of G.*
   (iii) *The element s normalizes the compact open subgroups K_m and inner conjugation by s takes the characters ψ_m to its inverse.*

Proof. Since U is generated by U_α, $\alpha \in \Phi$, it is enough to show that $\psi(sus^{-1}) = \psi^{-1}(u)$ for $u \in U_\alpha$. For $u = x_\alpha(\lambda) \in U_\alpha$ we have,

$$\begin{aligned}
\psi(sus^{-1}) &= \psi(sx_\alpha(\lambda)s^{-1})\\
&= \psi(x_\alpha(\alpha(s)\lambda))\\
&= \psi(x_\alpha(-\lambda))\\
&= \psi^{-1}(x_\alpha(\lambda))\\
&= \psi^{-1}(u).
\end{aligned}$$

For (ii), since $\alpha(s) = -1$ it is clear that $\alpha(s^2) = 1$ for all simple roots $\alpha \in \Delta$. Since $Z(G) = \bigcap_{\alpha \in \Delta} \mathrm{Ker}(\alpha)$, the result follows.

For (iii), It is easy to see that s normalizes U_m, $\bar{U}_m$. It follows that s normalizes G_m and hence K_m. Let $k = k^- k^\circ k^+ \in K_m$ (since $K_m = K_m^- K_m^\circ K_m^+$). We have

$$\begin{aligned}
\psi_m(sks^{-1}) &= \psi_m(sk^- s^{-1} sk^\circ s^{-1} sk^+ s^{-1})\\
&= \psi(sk^+ s^{-1})\\
&= \psi^{-1}(k^+)\\
&= \psi_m^{-1}(k).
\end{aligned}$$

$\qquad\square$

Theorem 4.3. *Let (π, W) be an irreducible smooth self-dual generic representation of G with non-zero vectors fixed under an Iwahori subgroup I in G. Suppose there exists an element $s \in T_\circ$ such that $\alpha(s) = -1$ for all simple roots α. Then $\varepsilon(\pi) = 1$.*

Proof. By Theorem 3.4, it is enough to show that $\omega_\pi(s^2) = 1$ (ω_π is the central character). Let $v \neq 0 \in \pi^I$. We have $v = \pi(s^2)v = \omega_\pi(s^2)v$. From this it follows that $\omega_\pi(s^2) = 1$. $\qquad\square$

Theorem 4.4. *There exists $s \in T_\circ$ such that $\alpha(s) = -1$ for all the simple roots α.*

Proof. We know that $X^\vee \otimes F^\times \simeq T$, via $y \otimes \lambda \mapsto y(\lambda)$. Since $F^\times \simeq \mathfrak{O}^\times \rtimes \mathbb{Z}$, we see that $T \simeq X^\vee \otimes \mathfrak{O}^\times \oplus X^\vee \otimes \mathbb{Z}$. Now define $f : T \longrightarrow \prod_{\alpha_i \in \Delta} F^\times$ by

$$f(y \otimes \lambda) = (\lambda^{\langle \alpha_1, y \rangle}, \dots, \lambda^{\langle \alpha_k, y \rangle}).$$

We will show that there exists $y \in X^\vee$ such that $\langle \alpha_i, y \rangle$ is an odd integer for every simple root $\alpha_i, i = 1, \dots, k$. Since Z is connected $X/\mathbb{Z}\Phi$ is torsion free. Since Δ spans Φ we see that $\mathbb{Z}\Phi = \mathbb{Z}\Delta$. Consider the exact sequence

(4.1) $$0 \longrightarrow \mathbb{Z}\Delta \longrightarrow X \longrightarrow X/\mathbb{Z}\Delta \longrightarrow 0.$$

Since (4.1) is an exact sequence of finitely generated free abelian groups, it is split. i.e., $X = \mathbb{Z}\Delta \oplus L$, where $L \simeq X/\mathbb{Z}\Delta$. Let $g \in \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}\Delta, \mathbb{Z})$.

Clearly, g extends to an element of $\text{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ (say trivial on L). Since $\text{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \simeq X^{\vee}$, there exists $y \in X^{\vee}$ such that $g = \langle -, y \rangle$. We now choose $h \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z})$ such that $h(\alpha_i)$ is odd for every $\alpha_i, i = 1, 2, \ldots, k$. Then $h(\alpha_i) = \langle \alpha_i, y \rangle$ is an odd integer. Now consider the element $y \otimes -1 \in X^{\vee} \otimes \mathfrak{O}^{\times}$. Let $s = y(-1)$. Then $s \in T_{\circ}$ clearly acts by -1 on all the simple root subgroups U_{α} of U, i.e.,

$$\begin{aligned}
sx_{\alpha_i}(\mu)s^{-1} &= x_{\alpha_i}(\alpha_i(s)\mu) \\
&= x_{\alpha_i}(\alpha_i(y(-1))\mu) \\
&= x_{\alpha_i}((-1)^{\langle \alpha_i, y \rangle}\mu) \\
&= x_{\alpha_i}(-\mu).
\end{aligned}$$

$\square$

4.2. Center of G is not connected.

4.2.1. *Construction of $(\tilde{G}, \tilde{T})$.* Let $q : X \to X/\mathbb{Z}\Phi$ be the canonical quotient map. Choose a free abelian group L of finite rank such that there exists a surjective map $p : L \to X/\mathbb{Z}\Phi$. Let p_1 and p_2 be the projection maps from $X \times L$ onto X and L respectively. Let

$$\tilde{X} = \{(x, l) \in X \times L \mid q(x) = p(l)\}.$$

Clearly $\tilde{X}$ is a free abelian group of finite rank. Let $\tilde{\Phi} = \{(\alpha, 0) \mid \alpha \in \Phi\}$. The map $\alpha \mapsto (\alpha, 0)$ induces an injection $\mathbb{Z}\Phi \hookrightarrow \tilde{X}$ and we identify its image under the map with $\mathbb{Z}\tilde{\Phi}$. Let $\tilde{X}^{\vee} = \text{Hom}_{\mathbb{Z}}(\tilde{X}, \mathbb{Z})$. Given $\tilde{\alpha} \in \tilde{\Phi}$ we want to describe $\tilde{\alpha}^{\vee} \in \tilde{\Phi}^{\vee} \subset \tilde{X}^{\vee}$. Now $\tilde{\alpha} = (\alpha, 0)$ for some $\alpha \in \Phi$. For this α, there exists $\alpha^{\vee} \in \Phi^{\vee} \subset X^{\vee} \simeq \text{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$. Let $\tilde{x} = (x, 0) \in \tilde{X}$. Define $\tilde{\alpha}^{\vee}(\tilde{x}) = \tilde{\alpha}^{\vee}((x, 0)) = \alpha^{\vee}(p_1(\tilde{x}))$. Clearly $\tilde{\alpha}^{\vee} \in \text{Hom}_{\mathbb{Z}}(\tilde{X}, \mathbb{Z})$. It is easy to see that $(\tilde{X}, \tilde{\Phi}, \tilde{X}^{\vee}, \tilde{\Phi}^{\vee})$ is a root datum. By the classification theorem for split groups, the existence of $(\tilde{G}, \tilde{T})$ follows. Since $\tilde{X}/\mathbb{Z}\tilde{\Phi} \hookrightarrow L$, it follows that it is torsion free and the center $\tilde{Z}$ of $\tilde{G}$ is connected.

4.2.2. *Extension of the central character.* In this section, we show that there exists a character ν of $\tilde{Z}$ which extends the central character ω_{π} and satisfies $\nu^2 = 1$.

Lemma 4.5. *There exists an unramified character $\mu : T \to \mathbb{C}^{\times}$ such that $\mu|_Z = \omega_{\pi}$.*

Proof. Since π has Iwahori fixed vectors, there exists an unramified character μ of T such that $\pi \hookrightarrow \text{Ind}_B^G \mu$. Let (ρ, E) be an irreducible sub-representation of $\text{Ind}_B^G \mu$ that is isomorphic to π. Let $x \in Z, f \in E, g \in G$. Clearly,

$$(4.2) \qquad (\rho(x)f)(g) = f(gx) = f(xg) = \mu(x)f(g)$$

On the other hand,

$$(4.3) \qquad (\rho(x)f)(g) = \omega_{\rho}(x)f(g)$$

From (4.2) and (4.3) it follows that $\omega_{\rho}(x) = \mu(x) = \omega_{\pi}(x)$. $\square$

Since μ is unramified it follows that $\mu|_Z = \omega_1$ (since $\mu|_{Z_\circ} = 1$). If we can extend ω_1 to a self-dual character $\tilde{\omega}_1$ of $\tilde{Z}_1$ then we get a self-dual character ν of $\tilde{Z}$ extending the central character ω_π. We record the result in a lemma below.

Lemma 4.6. *Suppose that $\tilde{\omega}_1$ is an extension of ω_1 to $\tilde{Z}_1$. Then there exists $\nu : \tilde{Z} \to \{\pm 1\}$ such that ν extends ω_π.*

Proof. For $\tilde{z} = \tilde{z}_0 \tilde{z}_1 \in \tilde{Z}$, $\tilde{z}_0 \in \tilde{Z}_\circ$, $\tilde{z}_1 \in \tilde{Z}_1$ define $\nu(\tilde{z}) = \tilde{\omega}_1(\tilde{z}_1)$. Clearly ν is a well-defined character of $\tilde{Z}$ and $\nu|_Z = \tilde{\omega}_1|_Z = \omega_1 = \mu|_Z = \omega_\pi$. $\square$

From Lemma 4.6, it follows that we can extend the central character ω_π to a self-dual character ν of $\tilde{Z}$ if there exists an extension $\tilde{\omega}_1$ of ω_1. Consider the map $\omega_1' : Z_1/Z_1^2 \to \{\pm 1\}$ defined by $\omega_1'(aZ_1^2) = \omega_1(a)$. Since Z_1/Z_1^2 is an elementary abelian 2-group, ω_1' can be thought of as a $\mathbb{Z}_2$-linear map. If the natural map from Z_1/Z_1^2 to $\tilde{Z}_1/\tilde{Z}_1^2$ is an embedding, then we can extend the $\mathbb{Z}_2$-linear map ω_1' to a $\mathbb{Z}_2$-linear map $\tilde{\omega}_1'$ of $\tilde{Z}_1/\tilde{Z}_1^2$. Now defining $\tilde{\omega}_1(a) = \tilde{\omega}_1'(a\tilde{Z}_2)$ gives us an extension of ω_1. The natural map is an embedding precisely when $\tilde{Z}_1^2 \cap Z_1 \subset Z_1^2$. We record the result in the following lemma.

Lemma 4.7. *The natural map Z_1/Z_1^2 to $\tilde{Z}_1/\tilde{Z}_1^2$ is an embedding and ω_1 extends to a character $\tilde{\omega}_1$ of $\tilde{Z}_1$.*

Proof. It is enough to show that $\tilde{Z}_1^2 \cap Z_1 \subset Z_1^2$. Consider $z_1 \in \tilde{Z}_1^2 \cap Z_1$. Clearly $z_1 = \tilde{z}_1^2$ for some $\tilde{z}_1 \in \tilde{Z}_1$. It is enough to show that $\tilde{z}_1 \in T$ (since $\tilde{z}_1 \in T$ implies $\tilde{z}_1 \in \tilde{Z}_1 \cap T = Z_1$ and $\tilde{z}_1^2 \in Z_1$). Since $T \hookrightarrow \tilde{T}$ there exists a sub-torus S such that $\tilde{T} = \tilde{T}_\circ \times \tilde{T}_1 = T \times S$. Clearly, $\tilde{T}_1 = T_1 \times S_1$. Indeed,

$$\begin{aligned}
\tilde{T} &= \tilde{T}_\circ \times \tilde{T}_1 \\
&= T_\circ \times T_1 \times S_\circ \times S_1 \\
&= T_\circ \times S_\circ \times T_1 \times S_1.
\end{aligned}$$

Now $\tilde{z}_1 \in \tilde{Z}_1 \subset \tilde{T}_1 = T_1 \times S_1$. Therefore $\tilde{z}_1 = t_1 s_1$ for $t_1 \in T_1$, $s_1 \in S_1$. Also $z_1 = \tilde{z}_1^2 = t_1^2 s_1^2 \in Z_1 \subset T_1$. We see that $s_1^2 = 1$. Since $\tilde{T}_1$ is torsion free it follows that $s_1 = 1$ and $\tilde{z}_1 \in T$. $\square$

4.2.3. *Irreducible representation $\tilde{\pi}$ of $\tilde{G}$.* In this section, we show that there exists an irreducible representation $\tilde{\pi}$ of $\tilde{G}$ which contains π with multiplicity one on restriction to G.

The main idea behind the proof is Theorem 3.2. We first extend the representation π to an irreducible representation $\pi\nu$ of $\tilde{Z}G$ and show that the group $\tilde{G}/\tilde{Z}G$ is finite abelian. Before we continue, we recall a result of Serre which we use in proving the finiteness of $\tilde{G}/\tilde{Z}G$.

Proposition 4.8 (Serre)**.** *If A is a finite Γ module, $H^n(\Gamma, A)$ is finite for every n.*

Proof. See Proposition 14, Sec. 5.1 in [11]. $\qquad\qquad\qquad\qquad\qquad$ $\square$

Let (π, W) be an irreducible representation of G and ν be a self-dual character of $\tilde{Z}$ extending the central character ω_π. Let $\pi\nu : \tilde{Z}G \to \mathrm{GL}(W)$ be defined as $(\pi\nu)(\tilde{z}g) = \nu(\tilde{z})\pi(g)$. Clearly, $\pi\nu$ is a well-defined irreducible representation of $\tilde{Z}G$. The irreducibility of $\pi\nu$ is trivial. To see $\pi\nu$ is well-defined, suppose $z_1 g_1 = z_2 g_2$. Then

$$
\begin{aligned}
(\pi\nu)(z_1 g_1) &= (\pi\nu)(z_1 z_2^{-1} z_2 g_1) \\
&= \nu(z_1 z_2^{-1}) \nu(z_2) \pi(g_1) \\
&= \omega_\pi(z_1 z_2^{-1}) \nu(z_2) \pi(g_1) \quad (\text{since } z_1 z_2^{-1} \in \tilde{Z} \cap G) \\
&= \omega_\pi(z_1 z_2^{-1}) \nu(z_2) \pi(z_1^{-1} z_2 g_2) \\
&= \omega_\pi(z_1 z_2^{-1}) \nu(z_2) \omega_\pi(z_1^{-1} z_2) \pi(g_2) \\
&= (\pi\nu)(z_2 g_2).
\end{aligned}
$$

We now prove the finiteness of $\tilde{G}/\tilde{Z}G$.

Theorem 4.9. $\tilde{G}/\tilde{Z}G$ *is a finite abelian group.*

Proof. Clearly, $\tilde{G} = \tilde{T}G$. Now

$$
\begin{aligned}
\tilde{G}/\tilde{Z}G &= \tilde{T}G/\tilde{Z}G \\
&= (\tilde{T}\tilde{Z}G)/\tilde{Z}G \\
&= \tilde{T}/(\tilde{T} \cap \tilde{Z}G) \\
&= \tilde{T}/\tilde{Z}T.
\end{aligned}
$$

It follows that $\tilde{G}/\tilde{Z}G$ is abelian. Let $\bar{F}$ be the algebraic closure of F and $\Gamma = \mathrm{Gal}(\bar{F}/F)$. Let $m : T(\bar{F}) \times \tilde{Z}(\bar{F}) \to \tilde{T}(\bar{F})$ be the multiplication map. This map is surjective with $\mathrm{Ker}(m) = \{(z, z^{-1}) \mid z \in Z(\bar{F})\}$ (follows by considering the dimensions). Considering $Z(\bar{F})$ embedded diagonally in $T(\bar{F}) \times \tilde{Z}(\bar{F})$, we get the following exact sequence of abelian groups

$$
1 \longrightarrow Z(\bar{F}) \longrightarrow T(\bar{F}) \times \tilde{Z}(\bar{F}) \xrightarrow{m} \tilde{T}(\bar{F}) \longrightarrow 1.
$$

Γ clearly acts on these groups and applying Galois cohomology, we get a long exact sequence of cohomology groups

$$
1 \longrightarrow Z(\bar{F})^\Gamma \longrightarrow T(\bar{F})^\Gamma \times \tilde{Z}(\bar{F})^\Gamma \longrightarrow \tilde{T}(\bar{F})^\Gamma \longrightarrow H^1(\Gamma, Z(\bar{F}))
$$
$$
\longrightarrow H^1(\Gamma, T(\bar{F}) \times \tilde{Z}(\bar{F})) \longrightarrow H^1(\Gamma, \tilde{Z}(\bar{F})) \longrightarrow \cdots
$$

We note that $H^1(\Gamma, T(\bar{F}) \times \tilde{Z}(\bar{F})) = 1$ (Hilbert 90) and we get the short exact sequence

(4.4) $\qquad 1 \longrightarrow Z \longrightarrow T \times \tilde{Z} \xrightarrow{m} \tilde{T} \xrightarrow{\varphi} H^1(\Gamma, Z(\bar{F})) \longrightarrow 1.$

From (4.4) it follows that φ is surjective, $\mathrm{Im}(m) = T\tilde{Z} = \mathrm{Ker}(\varphi)$, and $\tilde{T}/\tilde{Z}T \simeq H^1(\Gamma, Z(\bar{F}))$. It is enough to show that $H^1(\Gamma, Z(\bar{F}))$ is finite. Let Z° be the identity component of the algebraic group Z. Consider the short exact sequence

(4.5) $\qquad 1 \longrightarrow Z^\circ(\bar{F}) \longrightarrow Z(\bar{F}) \longrightarrow Z(\bar{F})/Z^\circ(\bar{F}) \longrightarrow 1.$

Applying Galois cohomology again to (4.5), we get the sequence

$$1 \longrightarrow Z^\circ \longrightarrow Z \longrightarrow Z/Z^\circ \longrightarrow H^1(\Gamma, Z^\circ(\bar{F})) \longrightarrow H^1(\Gamma, Z(\bar{F}))$$

$$\longrightarrow H^1(\Gamma, Z(\bar{F})/Z^\circ(\bar{F})) \longrightarrow \cdots$$

Since $Z^\circ(\bar{F})$ is connected, we have $H^1(\Gamma, Z^\circ(\bar{F})) = 1$ and it follows that $H^1(\Gamma, Z(\bar{F})) \hookrightarrow H^1(\Gamma, Z(\bar{F})/Z^\circ(\bar{F}))$. Since F is a local field of char 0 and $Z(\bar{F})/Z^\circ(\bar{F})$ is a finite abelian group, $H^1(\Gamma, Z(\bar{F})/Z^\circ(\bar{F}))$ is finite (Proposition 4.8). Hence the result follows. $\qquad \square$

By Theorem 3.1 and Theorem 3.2, we get an irreducible representation $(\tilde{\pi}, V)$ of $\tilde{G}$ which breaks up as a finite direct sum of distinct irreducible representations $\pi_1, \ldots, \pi_k$ each occurring with the same multiplicity m on restriction to $\tilde{Z}G$ and contains $\pi\nu$ as a constituent. Without loss of generality, we assume that $\pi_1 \simeq \pi\nu$. To simplify notation, we again denote the restriction of π_i's to G by π_i so that $\tilde{\pi}|_G = m\pi_1 \oplus m\pi_2 \oplus \ldots \oplus m\pi_k$ and $\pi \simeq \pi_1$. We now show that each π_i occurs with multiplicity one in $\tilde{\pi}|_G$.

Lemma 4.10. *The representation $(\tilde{\pi}, V)$ of $\tilde{G}$ is generic and each irreducible representation π_i occurs with multiplicity one.*

Proof. Since (π, W) is generic, there exists a non-degenerate character ψ of U such that $\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \psi) \neq 0$. It is enough to show that $\mathrm{Hom}_{\tilde{G}}(\tilde{\pi}, \mathrm{Ind}_U^{\tilde{G}} \psi) \neq 0$. Observe that $\tilde{U} = U \subset G$. Consider the restriction $\tilde{\pi}|_U$ of $\tilde{\pi}$. Since π is generic, $\mathrm{Hom}_U(\pi|_U, \psi) \neq 0$. It follows that

$$\mathrm{Hom}_U(\tilde{\pi}|_U, \psi) \simeq \mathrm{Hom}_{\tilde{G}}(\tilde{\pi}, \mathrm{Ind}_U^{\tilde{G}} \psi) \neq 0.$$

Indeed,

$$\begin{aligned} \mathrm{Hom}_U(\tilde{\pi}|_U, \psi) &= \mathrm{Hom}_U((\tilde{\pi}|_G)|_U, \psi) \\ &= \mathrm{Hom}_U((m\pi_1 \oplus m\pi_2 \oplus \cdots \oplus m\pi_k)|_U, \psi) \\ &= m \bigoplus_{i=1}^{k} \mathrm{Hom}_U(\pi_i|_U, \psi) \\ &\neq 0. \end{aligned}$$

As $\tilde{\pi}$ is generic it follows that $dim(\mathrm{Hom}_{\tilde{G}}(\tilde{\pi}, \mathrm{Ind}_U^{\tilde{G}} \psi)) = 1$. Thus by Frobenius Reciprocity, $m = 1$. $\qquad \square$

4.2.4. *Choosing $\tilde{\pi}$ with non-zero $\tilde{I}$ fixed vectors.* In this section, we show that the representation $\tilde{\pi}$ can be modified in such a way that it has non-zero vectors fixed under an Iwahori subgroup $\tilde{I}$ in $\tilde{G}$.

Lemma 4.11. *Suppose that τ_1 is a linear character of $\tilde{I}$ which is trivial on I. Then τ_1 extends to a linear character $\tilde{\tau}$ of $\tilde{G}$ which is trivial on G.*

Proof. Let $I^- = I \cap \bar{U}$ and $I^+ = I \cap U$. We know that $I = I^- T_\circ I^+$. Since $U = \tilde{U}$ we have $\tilde{I} = I^- \tilde{T}_\circ I^+$. Now

$$\tilde{I}/I = \tilde{T}_\circ I/I \ (\tilde{T}_\circ \text{ normalizes } I^-, I^+)$$
$$= \tilde{T}_\circ/\tilde{T}_\circ \cap I$$
$$= \tilde{T}_\circ/T_\circ.$$

It follows that we can consider τ_1 as a linear character of $\tilde{T}_\circ$ which is trivial on $T_\circ$. We first extend τ_1 to a character $\tilde{\tau}_1$ of $\tilde{T}$ by making it trivial on $\tilde{T}_1$, i.e., $\tilde{\tau}_1(\tilde{t}_\circ \tilde{t}_1) = \tau_1(\tilde{t}_\circ)$. Now define an extension $\tilde{\tau}$ of $\tilde{\tau}_1$ to $\tilde{G}$ as $\tilde{\tau}(\tilde{t}g) = \tilde{\tau}_1(\tilde{t}), \tilde{t} \in \tilde{T}, g \in G$ (this is possible since $\tilde{G} = \tilde{T}G$). Using

$$\tilde{G}/G = \tilde{T}G/T = \tilde{T}/T = \tilde{T}_\circ/T_\circ \times \tilde{T}_1/T_1$$

it follows that $\tilde{\tau}$ is well-defined and a character of $\tilde{G}$. $\qquad\square$

Theorem 4.12. *The representation $(\tilde{\pi}\tau^{-1}, V)$ of $\tilde{G}$ has non-zero $\tilde{I}$ fixed vectors.*

Proof. Let $v \neq 0 \in V$ be such that $\tilde{\pi}(i)v = v, \forall i \in I$ (v exists since $\tilde{\pi}|_G \supset \pi$ and π has non-zero vectors fixed under I). Let $V_0 = \text{Span}_{\mathbb{C}}\{\tilde{\pi}(k)v \mid k \in \tilde{I}\}$. Clearly, V_0 is an invariant subspace for $\tilde{I}$ and thus we get a representation (ρ, V_0) of $\tilde{I}$. Suppose $\rho = \tau_1 \oplus \cdots \oplus \tau_k$, where each τ_i is an irreducible representation of $\tilde{I}$. We know that $1_I \subset \rho$. Pick an irreducible component, say τ_1, that contains 1_I. By Clifford's theorem, $I \leq \text{Ker}(\tau_1)$. Since $\tilde{I}/I$ is a compact abelian group, it follows that τ_1 is a linear character of $\tilde{I}$ which is trivial on I. By Lemma 4.11, τ_1 extends to a linear character $\tilde{\tau}$ of $\tilde{G}$ trivial on G. Consider the irreducible representation $\tilde{\pi}\tau^{-1}$. Clearly it has an $\tilde{I}$ fixed vector. Indeed for w in the space of τ_1 and $k \in \tilde{I}$, we have

$$(\tilde{\pi}\tau^{-1})(k)w = \tau^{-1}(k)\tilde{\pi}(k)w$$
$$= \tau^{-1}(k)\tau_1(k)w$$
$$= w \ (\text{ since } \tau|_{\tilde{I}} = \tau_1).$$

$\qquad\square$

It is easy to see that $\tilde{\pi}\tau^{-1}|_G$ contains the representation π with multiplicity one, in addition to having non-zero $\tilde{I}$ fixed vectors. To simplify notation, we will denote the representation $\tilde{\pi}\tau^{-1}$ as $\tilde{\pi}$.

4.2.5. *Sign of $\tilde{\pi}$.* In this section, we attach a sign $\varepsilon(\tilde{\pi})$ to the representation $\tilde{\pi}$. We also give a formula to compute $\varepsilon(\tilde{\pi})$ and show that $\varepsilon(\tilde{\pi}) = \varepsilon(\pi)$. Finally we show that $\varepsilon(\tilde{\pi}) = 1$ to complete the proof of the main theorem.

Consider the representation $\tilde{\pi}^\vee$. This is again an irreducible representation of $\tilde{G}$ which on restriction to $\tilde{Z}G$ and contains the representation $\pi\nu$ with multiplicity 1 (since $\nu = \nu^{-1}$ and $\pi \simeq \pi^\vee$). By Theorem 3.2, there's a linear character χ of $\tilde{G}$ trivial on $\tilde{Z}G$ such that $\tilde{\pi}^\vee \simeq \tilde{\pi} \otimes \chi$. We use this isomorphism to define a non-degenerate bilinear form $[\,,\,]$ on V.

Lemma 4.13. *There exists a non-degenerate form $[\,,\,] : V \times V \to \mathbb{C}$ satisfying $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] = \chi^{-1}(g)[v_1, v_2]$.*

Proof. Since $\tilde{\pi}^\vee \simeq \tilde{\pi} \otimes \chi$, there exists a non-zero map $\phi : V \to V^\vee$ such that $\tilde{\pi}^\vee(g)(\phi(v)) = \phi((\tilde{\pi} \otimes \chi)(g)v)$. Let $\langle\,,\,\rangle : V \times V^\vee \to \mathbb{C}$ be the canonical $\tilde{G}$ invariant pairing. We define $[\,,\,] : V \times V \to \mathbb{C}$ as $[v_1, v_2] = \langle v_1, \phi(v_2)\rangle$. Clearly this form is non-degenerate and satisfies $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] = \chi^{-1}(g)[v_1, v_2]$. Indeed,

$$\begin{aligned}
[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] &= \langle \tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2\rangle \\
&= \langle \tilde{\pi}(g)v_1, \chi^{-1}(g)\tilde{\pi}^\vee(g)(\phi(v_2))\rangle \\
&= \chi^{-1}(g)\langle \tilde{\pi}(g)v_1, \tilde{\pi}^\vee(g)(\phi(v_2))\rangle \\
&= \chi^{-1}(g)\langle v_1, \phi(v_2)\rangle \\
&= \chi^{-1}(g)[v_1, v_2].
\end{aligned}$$

$\square$

The form $[\,,\,]$ is unique up to scalars and is easily seen to be symmetric or skew-symmetric as before, i.e.,

$$[v_1, v_2] = \varepsilon(\tilde{\pi})[v_2, v_1].$$

where $\varepsilon(\tilde{\pi}) \in \{\pm 1\}$. We call $\varepsilon(\tilde{\pi})$ the sign of $\tilde{\pi}$.

Let $[\,,\,] : V \times V \longrightarrow \mathbb{C}$ be the non-degenerate bilinear form on V (obtained above). Suppose that $[\,,\,]\big|_{W_1 \times W_j} = 0, \forall j = 2, 3, \cdots, k$, then it is easy to see that $[\,,\,]\big|_{W_1 \times W_1}$ is non-degenerate. We now show that $[\,,\,]\big|_{W_1 \times W_j} = 0$ for $j = 2, 3, \cdots, k$.

Lemma 4.14. $[\,,\,]\big|_{W_1 \times W_j} = 0, \ \forall j = 2, 3, \cdots, k$.

Proof. Suppose $[\,,\,]\big|_{W_1 \times W_j} \neq 0$. Let $v \in W_1$ and $u \in W_j$ be such that $[v, u] \neq 0$. Let $\phi(w) = \phi_w$ be defined as $\phi_w(v) = [v, w]$. Then clearly ϕ is a

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non-zero intertwining map between $\pi_1^\vee$ and π_j. Indeed,

$$
\begin{aligned}
(\pi_1^\vee(g) \circ \phi)(w_j)(w_1) &= \pi_1^\vee(g)(\phi(w_j)(w_1) \\
&= \phi(w_j)(\pi_1(g^{-1})w_1) \\
&= [\pi_1(g^{-1})w_1, w_j] \\
&= [w_1, \pi_j(g)w_j] \\
&= \phi(\pi_j(g)w_j)(v_1).
\end{aligned}
$$

Since $\pi_1^\vee \simeq \pi_1$ and the representations π_i are distinct (up to isomorphism), the lemma follows. $\qquad\square$

We know that $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] = \chi^{-1}(g)[v_1, v_2], \forall g \in \tilde{G}, v_1, v_2 \in V$. Now if $g \in G$ then $\chi(g) = 1$ and we have $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] = [v_1, v_2]$. In particular if $v_1 \in W_1$ and $v_j \in W_j$, then $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_j] = [v_1, v_j]$. Since $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k, \tilde{\pi}(g)v_1 = \pi_1(g)v_1, \tilde{\pi}(g)v_j = \pi_j(g)v_j$.

Lemma 4.15. *With notation as above, $\varepsilon(\tilde{\pi}) = \varepsilon(\pi)$.*

Proof. Since $\chi(g) = 1$ for $g \in G$, we have

$$
\begin{aligned}
[\tilde{\pi}(g)w_1, \tilde{\pi}(g)\dot{w}_1] &= [\pi_1(g)w_1, \pi_1(g)\dot{w}_1] \\
&= [w_1, \dot{w}_1].
\end{aligned}
$$

Since $\pi_1 \simeq \pi$, we see that $[\,,\,]\big|_{W_1 \times W_1}$ is G-invariant. Therefore $[w_1, \dot{w}_1] = \varepsilon(\pi)(w_1, \dot{w}_1)$. But we also know that $[w_1, \dot{w}_1] = \varepsilon(\tilde{\pi})[\dot{w}_1, w_1]$. $\qquad\square$

By Lemma 4.15, it follows that the sign $\varepsilon(\pi)$ is completely determined by the sign $\varepsilon(\tilde{\pi})$. Since $\tilde{Z}$ is connected, applying Theorem 4.4 we get an element $s \in \tilde{T}_\circ$ such that $\alpha(s) = -1$ for all simple roots α of $\tilde{T}$. We will show that the sign $\varepsilon(\tilde{\pi})$ is controlled by the central character $\omega_{\tilde{\pi}}$ and the character χ. Before we proceed, we prove a lemma we need.

Lemma 4.16. *Let W_1, W_2 be irreducible K-invariant subspaces of V. Let $\rho_1 = \tilde{\pi}|_{W_1}$ and $\rho_2 = \tilde{\pi}|_{W_2}$. Let $b : W_2 \to W_1^\vee$ be the map $w_2 \mapsto [-, w_2]$. If $b \neq 0$, then $\rho_2 \simeq \rho_1^\vee \chi$.*

Proof. We first show that b defines an intertwining map between ρ_2 and $\rho_1^\vee \chi$. Indeed, for $w_1 \in W_1, w_2 \in W_2$ and $k \in K$, we have

$$
\begin{aligned}
b(\rho_2(k)(w_2))(w_1) &= [w_1, \rho_2(k)w_2] \\
&= \chi(k)[\rho_1(k^{-1})(w_1), w_2] \\
&= \chi(k)b(w_2)(\rho_1(k^{-1})w_1) \\
&= \chi(k)\rho_1^\vee(k)(b(w_2))(w_1).
\end{aligned}
$$

Since $b \neq 0$, Schur's Lemma applies and the result follows. $\qquad\square$

Let V_0 be the space of $\psi_{\tilde{K}}$ and $v_0 \in V_0$. Since $[\,,\,]$ is non-degenerate, $b(v_0)(v_1) = [v_1, v_0] \neq 0$ for some $v_1 \in V_1$ where V_1 is an irreducible $\tilde{K}$-invariant subspace of V. We denote ρ for the restriction of $\tilde{\pi}|_{V_1}$. By

Lemma 4.16, it follows that $\rho \simeq \psi_{\tilde{K}}^{-1} \chi$. Since χ is smooth, we can in fact choose $\tilde{K}$ such that χ is trivial on $\tilde{K}$. It follows that any vector $v_0 \in V_0$ has to pair non-trivially with some vector in the space of $\psi_{\tilde{K}}^{-1}$, i.e., $[\tilde{\pi}(s)v_0, v_0] \neq 0$. We will use this in the following theorem.

Theorem 4.17. *Let $(\tilde{\pi}, V)$ be the irreducible representation of $\tilde{G}$ obtained above. Then $\varepsilon(\tilde{\pi}) = \omega_{\tilde{\pi}}(s^2)\chi(s)$.*

Proof. Clearly $s^2 \in \tilde{Z}$. Since $\tilde{\pi}$ is generic it follows by Theorem 3.5 that there exists a compact open subgroup $\tilde{K}$ and a character $\psi_{\tilde{K}}$ of $\tilde{K}$ such that $\psi_{\tilde{K}}$ occurs with multiplicity one in $\tilde{\pi}|_{\tilde{K}}$. Let V_0 be the space of $\psi_{\tilde{K}}$ and $0 \neq v_0 \in V_0$. Now

$$\begin{aligned}
[\tilde{\pi}(s)v_0, \tilde{\pi}(s^2)v_0] &= \omega_{\tilde{\pi}}(s^2)[\tilde{\pi}(s)v_0, v_0] \quad \text{(since } s^2 \in \tilde{Z}) \\
&= \chi^{-1}(s)[v_0, \tilde{\pi}(s)v_0] \quad \text{(invariance of the form)}
\end{aligned}$$

It follows that $\varepsilon(\tilde{\pi}) = \omega_{\tilde{\pi}}(s^2)\chi(s)$. $\qquad\qquad\square$

Using $\tilde{\pi}$ has Iwahori fixed vectors and $s^2 \in \tilde{T}_\circ$, we see that $\omega_{\tilde{\pi}}(s^2) = 1$. It will follow that $\varepsilon(\pi) = 1$ once we show that $\chi(s) = 1$. We do this by showing that χ is an unramified character. Before we continue, we recall a result about intertwining maps which we need in the proof.

For K a compact open subgroup of $\tilde{G}$ and ρ an irreducible representation of K, we let $\hat{K}$ denote the set of equivalence classes of irreducible smooth representations of K, $K^g = g^{-1}Kg$, $g \in \tilde{G}$ and ρ^g the irreducible representation of K^g defined as $x \to \rho(gxg^{-1})$.

Proposition 4.18. *For $i = 1, 2$, let K_i be a compact open subgroup of $\tilde{G}$ and let $\rho_i \in \hat{K}_i$. Let (Π, V) be an irreducible representation of $\tilde{G}$ which contains both ρ_1 and ρ_2. There then exists $g \in \tilde{G}$ such that $\mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2) \neq 0$.*

Proof. We refer the reader to [3] (Chapter 3, Section 11, Proposition 1) for a proof of the above proposition. In [3], the authors prove the result for $\mathrm{GL}(2, F)$. The same proof works in the case of any connected reductive group. $\qquad\qquad\square$

Theorem 4.19. *The character χ is an unramified character. In particular $\chi(s) = 1$.*

Proof. We know that $\tilde{\pi}^\vee \simeq \tilde{\pi} \otimes \chi$. Since $\tilde{\pi}$ has non-trivial $\tilde{I}$ fixed vectors it follows that $\tilde{\pi}^\vee$ and hence $\tilde{\pi} \otimes \chi$ has non-trivial $\tilde{I}$ fixed vectors. Therefore $(\tilde{\pi} \otimes \chi)\big|_{\tilde{I}} \supset 1$ and $(\tilde{\pi} \otimes \chi)\big|_{\tilde{I}} \supset \chi$. By Proposition 4.18 there exists $g \in \tilde{G}$ such that $\mathrm{Hom}_{\tilde{I}^g \cap \tilde{I}}(1^g, \chi) \neq 0$. Since $\tilde{G} = \coprod_{w \in \tilde{W}} \tilde{I}w\tilde{I}$ we see that $\mathrm{Hom}_{\tilde{I}^w \cap \tilde{I}}(1^w, \chi) \neq 0$ when $g \in \tilde{I}w\tilde{I}$. From this it follows that $\chi(h) = 1, \forall h \in \tilde{I}^w \cap \tilde{I}$. Since $\tilde{T}_\circ \subset \tilde{I}^w \cap \tilde{I}$ it follows that χ is unramified. $\qquad\qquad\square$

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